

SOME 3-ADIC CONGRUENCES FOR BINOMIAL SUMS

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ABSTRACT. We prove some 3-adic congruences for binomial sums, which were conjectured by Sun.

1. INTRODUCTION

For a non-zero integer n and a prime p , let $\nu_p(n)$ denote the p -adic order of n , i.e., $\nu_p(n)$ is the largest integer such that $p^{\nu_p(n)} \mid n$. In [1], Strauss, Shallit and Zagier proved that for any positive integer n ,

$$\nu_3\left(\sum_{k=0}^{n-1} \binom{2k}{k}\right) = 2\nu_3(n) + \nu_3\left(\binom{2n}{n}\right). \quad (1.1)$$

Recently Sun [1] showed that

$$\nu_p\left(\sum_{k=0}^{n-1} \frac{1}{m^k} \binom{2k}{k}\right) \geq \nu_p(n) \quad \text{and} \quad \nu_p\left(\sum_{k=0}^{n-1} \frac{(-1)^k}{m^k} \binom{2k}{k} \binom{n-1}{k}\right) \geq \nu_p(n), \quad (1.2)$$

where m is an integer and p is an odd prime dividing $m-4$. Furthermore, he also proposed several conjectures on the 3-adic orders of the above two binomial sums.

Conjecture 1.1. *Let m be integer with $m \equiv 1 \pmod{3}$.*

(i) *For every positive integer n , we have*

$$\nu_3\left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{m^k}\right) \geq \min\{\nu_3(n), \nu_3(m-1) - 1\}. \quad (1.3)$$

For any integer $a \geq \nu_3(m-1)$, we have

$$\frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{1}{m^k} \binom{2k}{k} \equiv \frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}. \quad (1.4)$$

(ii) *For every positive integer n , we have*

$$\nu_3\left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{(-1)^k}{m^k} \binom{n-1}{k} \binom{2k}{k}\right) \geq \min\{\nu_3(n), \nu_3(m-1)\} - 1. \quad (1.5)$$

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For any integer $a > \nu_3(m-1)$, we have

$$\frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{(-1)^k}{m^k} \binom{3^a-1}{k} \binom{2k}{k} \equiv -\frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}. \quad (1.6)$$

(iii) For any integer $a \geq 2$, we have

$$\frac{1}{3^a} \sum_{k=0}^{3^a-1} (-1)^k \binom{3^a-1}{k} \binom{2k}{k} \equiv -3^{a-1} \pmod{3^a}. \quad (1.7)$$

In this paper, we shall confirm Conjecture.

Theorem 1.1. *All assertions of Conjecture 1.1 are true.*

The proofs of (1.3)-(1.7) will be given in the next sections.

2. PROOFS OF (1.3) AND (1.4)

For $A, B \in \mathbb{Z}$, the Lucas sequence $\{u_n(A, B)\}$ are given by

$$u_0(A, B) = 0, \quad u_1(A, B) = 1, \quad u_{n+1}(A, B) = Au_n(A, B) - Bu_{n-1}(A, B) \text{ for } n \geq 1.$$

In particular, it is not difficult to check that $\{u_n(-1, 1)\}_{n \geq 0} \in \{0, 1, -1\}$ and $u_n(-1, 1) \equiv n \pmod{3}$.

Lemma 2.1. *Suppose that $m \equiv 1 \pmod{3}$. Then we have*

$$\frac{u_n(m-2, 1)}{n} \equiv \frac{u_n(-1, 1)}{n} + \frac{m-1}{3} \binom{n-1}{2} \pmod{3^{\nu_3(m-1)}} \quad (2.1)$$

if $m \neq 4$, and

$$\frac{u_n(2, 1)}{n} \equiv \frac{u_n(-1, 1)}{n} \pmod{3}. \quad (2.2)$$

In particular, we always have

$$\frac{u_n(m-2, 1)}{n} \equiv \frac{u_n(-1, 1)}{n} \pmod{3^{\nu_3(m-1)-1}}. \quad (2.3)$$

Proof. Let $\Delta = m(m-4)$. By the properties of Lucas sequences, we have

$$u_n(m-2, 1) = \frac{1}{2^{n-1}} \sum_{\substack{1 \leq k \leq n \\ 2 \nmid k}} \frac{n}{k} \binom{n-1}{k-1} (m-2)^{n-k} \Delta^{(k-1)/2}. \quad (2.4)$$

If $\Delta = 0$, i.e., $m = 4$, then

$$\frac{u_n(2, 1)}{n} = \left(\frac{4-2}{2} \right)^{n-1} = 1 \equiv \frac{u_n(-1, 1)}{n} \pmod{3}.$$

Suppose that $\Delta \neq 0$. Then

$$\begin{aligned}
& \frac{u_n(m-2, 1)}{n} - \left(\frac{m-2}{2}\right)^{n-1} = \sum_{\substack{1 \leq k \leq n \\ 2 \nmid k}} \binom{n-1}{k-1} \left(\frac{m-2}{2}\right)^{n-k} \frac{\Delta^{(k-1)/2}}{k2^{k-1}} \\
&= \sum_{\substack{1 \leq k \leq n \\ 2 \nmid k}} \binom{n-1}{k-1} \frac{(m-2)^{n-k}}{2^{n-k}} \cdot \frac{((m-1)(m-3)-3)^{(k-1)/2}}{k2^{k-1}} \\
&= \sum_{\substack{1 \leq k \leq n \\ 2 \nmid k}} \binom{n-1}{k-1} \frac{(m-2)^{n-k}}{k2^{n-1}} \sum_{j=0}^{(k-1)/2} \binom{(k-1)/2}{j} (m-1)^j (m-3)^j (-3)^{(k-1)/2-j} \\
&\equiv \sum_{\substack{1 \leq k \leq n \\ 2 \nmid k}} \binom{n-1}{k-1} \left(-\frac{1}{2}\right)^{n-k} \frac{(-3)^{(k-1)/2}}{k2^{k-1}} + \frac{m-1}{3} \binom{n-1}{2} \pmod{3^{\nu_3(m-1)}}.
\end{aligned}$$

By (2.4), it is derived that

$$\begin{aligned}
& \sum_{\substack{1 \leq k \leq n \\ 2 \nmid k}} \binom{n-1}{k-1} \left(-\frac{1}{2}\right)^{n-k} \frac{(-3)^{(k-1)/2}}{k2^{k-1}} \\
&= \frac{u_n(1-2, 1)}{n} - \left(-\frac{1}{2}\right)^{n-1} \equiv \frac{u_n(m-2, 1)}{n} - \left(\frac{m-2}{2}\right)^{n-1} \pmod{3^{\nu_3(m-1)}}.
\end{aligned}$$

We are done. \square

The following curious identity is due to Sun and Taurso [3, (2.1)]:

$$m^{n-1} \sum_{k=0}^{n-1} \frac{1}{m^k} \binom{2k}{k} = \sum_{k=0}^{n-1} \binom{2n}{k} u_{n-k}(m-2, 1). \quad (2.5)$$

It is easy to check that

$$\frac{1}{n} \binom{2n}{k} = \frac{1}{n-k} \left(2 \binom{2n-1}{k} - \binom{2n}{k} \right). \quad (2.6)$$

So (2.5) can be rewritten as

$$\frac{m^{n-1}}{n} \sum_{k=0}^{n-1} \frac{1}{m^k} \binom{2k}{k} = \sum_{k=0}^{n-1} \left(2 \binom{2n-1}{k} - \binom{2n}{k} \right) \frac{u_{n-k}(m-2, 1)}{n-k}. \quad (2.7)$$

Thus using (2.3) and (2.6), we have

$$\begin{aligned} \frac{m^{n-1}}{n} \sum_{k=0}^{n-1} \frac{1}{m^k} \binom{2k}{k} &\equiv \sum_{k=0}^{n-1} \left(2 \binom{2n-1}{k} - \binom{2n}{k} \right) \frac{u_{n-k}(-1, 1)}{n-k} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \binom{2n}{k} u_{n-k}(-1, 1) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{2k}{k} \pmod{3^{\nu_3(m-1)-1}}. \end{aligned}$$

Thus (1.3) easily follows from (1.1).

Suppose that $a \geq \nu_3(m-1)$. When $m = 4$, we have $u_n(2, 1) = n$. By (2.5),

$$\begin{aligned} \frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{1}{4^k} \binom{2k}{k} &= \frac{1}{3^a \cdot 4^{3^a-1}} \sum_{k=0}^{3^a-1} \binom{2 \cdot 3^a}{k} (3^a - k) \\ &= \frac{1}{4^{3^a-1}} \sum_{k=0}^{3^a-1} \binom{2 \cdot 3^a}{k} - \frac{2}{4^{3^a-1}} \sum_{k=1}^{3^a-1} \binom{2 \cdot 3^a - 1}{k-1} \\ &\equiv \sum_{k=0}^{3^a-1} \binom{2 \cdot 3^a}{k} + \sum_{k=1}^{3^a-1} \binom{2 \cdot 3^a - 1}{k-1} \equiv 1 \pmod{3}. \end{aligned}$$

Suppose that $m \neq 4$. Note that

$$\begin{aligned} &\sum_{k=0}^{3^a-1} \left(2 \binom{2 \cdot 3^a - 1}{k} - \binom{2 \cdot 3^a}{k} \right) \cdot \frac{u_{3^a-k}(-1, 1)}{3^a - k} \\ &= \frac{1}{3^a} \sum_{k=0}^{3^a-1} \binom{2 \cdot 3^a}{k} u_{3^a-k}(-1, 1) = \frac{1}{3^a} \sum_{k=0}^{3^a-1} \binom{2k}{k} \equiv 0 \pmod{3^a}. \end{aligned}$$

Hence applying (2.1) and (2.7), we get

$$\begin{aligned} \frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{1}{m^k} \binom{2k}{k} &= \frac{1}{m^{3^a-1}} \sum_{k=0}^{3^a-1} \left(2 \binom{2 \cdot 3^a - 1}{k} - \binom{2 \cdot 3^a}{k} \right) \cdot \frac{u_{3^a-k}(m-2, 1)}{3^a - k} \\ &\equiv \sum_{k=0}^{3^a-1} \left(2 \binom{2 \cdot 3^a - 1}{k} - \binom{2 \cdot 3^a}{k} \right) \cdot \frac{m-1}{3} \binom{3^a - k - 1}{2} \\ &\equiv \frac{m-1}{3} \left(\sum_{k=0}^{3^a-1} 2 \binom{2 \cdot 3^a - 1}{k} \binom{2 \cdot 3^a - k - 1}{2} - \binom{3^a - 1}{2} \right) \\ &= \frac{m-1}{3} \left(\sum_{k=0}^{3^a-1} (2 \cdot 3^a - 1)(2 \cdot 3^a - 2) \binom{2 \cdot 3^a - 3}{k} - \binom{3^a - 1}{2} \right) \\ &\equiv \frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}, \end{aligned}$$

where in the last step we use the fact

$$\sum_{k=0}^{3^a-1} \binom{2 \cdot 3^a - 3}{k} = 2^{2 \cdot 3^a - 4} + \frac{1}{2} \binom{2 \cdot 3^a - 3}{3^a - 2} + \binom{2 \cdot 3^a - 3}{3^a - 1} \equiv 1 \pmod{3}.$$

This proves (1.4).

3. PROOFS OF (1.5) AND (1.7)

In this section, we shall prove (1.5) and (1.6), under the assumption of the following congruence:

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \binom{2k}{k} \equiv 0 \pmod{3^{2\nu_3(n)-1}}. \quad (3.1)$$

And the proof of (3.1) will be given in the final section. We also need a special case of an identity of Sun [2, (2.6)]:

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{(-1)^k}{m^k} \binom{n-1}{k} \binom{2k}{k} = \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \binom{n-1}{k-1} \sum_{l=0}^{k-1} \frac{1}{m^l} \binom{2l}{l}. \quad (3.2)$$

Thus from (2.3), (2.7) and (3.1), it follows that

$$\begin{aligned} & \frac{1}{n} \sum_{k=0}^{n-1} \frac{(-1)^k}{m^k} \binom{n-1}{k} \binom{2k}{k} \\ &= \sum_{k=1}^n \frac{(-1)^{k-1}}{m^{k-1}} \binom{n-1}{k-1} \sum_{l=0}^{k-1} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \frac{u_{k-l}(m-2, 1)}{k-l} \\ &\equiv \sum_{k=1}^n \frac{(-1)^{k-1}}{m^{k-1}} \binom{n-1}{k-1} \sum_{l=0}^{k-1} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \frac{u_{k-l}(-1, 1)}{k-l} \\ &\equiv \frac{1}{n} \sum_{k=0}^{n-1} \frac{(-1)^{k-1}}{m^{k-1}} \binom{n-1}{k} \binom{2k}{k} \equiv 0 \pmod{3^{\nu_3(m-1)-1}}. \end{aligned}$$

So (1.5) is concluded.

First, suppose that $m \neq 4$. For an integer $a \geq \nu_3(m-1) + 1$, define

$$f(a) = \sum_{k=1}^{3^a} \frac{(-1)^{k-1}}{m^{k-1}} \binom{3^a-1}{k-1} \sum_{l=0}^{k-1} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \binom{k-l-1}{2}.$$

By (2.1),

$$\begin{aligned}
& \sum_{k=1}^{3^a} \frac{(-1)^{k-1}}{m^{k-1}} \binom{3^a-1}{k-1} \sum_{l=0}^{k-1} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \frac{u_{k-l}(m-2, 1)}{k-l} \\
& \equiv \sum_{k=1}^{3^a} \frac{(-1)^{k-1}}{m^{k-1}} \binom{3^a-1}{k-1} \sum_{l=0}^{k-1} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \frac{u_{k-l}(-1, 1)}{k-l} \\
& \quad + \sum_{k=1}^{3^a} \frac{(-1)^{k-1}}{m^{k-1}} \binom{3^a-1}{k-1} \sum_{l=0}^{k-1} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \cdot \frac{m-1}{3} \binom{k-l-1}{2} \\
& = \frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{(-1)^{k-1}}{m^{k-1}} \binom{3^a-1}{k} \binom{2k}{k} + \frac{m-1}{3} f(a) \pmod{3^{\nu_3(m-1)}}.
\end{aligned}$$

Thus in view of (3.1), it suffices to show that

$$f(a) \equiv -1 \pmod{3}.$$

Noting that

$$\binom{k-l-1}{2} = \frac{(k-l-1)(k-l-2)}{2} \equiv \begin{cases} 1 \pmod{3}, & \text{if } 3 \mid k-l, \\ 0 \pmod{3}, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}
& \sum_{l=0}^{k-1} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \binom{k-l-1}{2} \\
& \equiv \sum_{\substack{0 \leq j \leq k-l \\ 3 \mid k-l}} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \pmod{3}.
\end{aligned}$$

By the proof of [2, Theorem 1.1],

$$\sum_{\substack{0 \leq j \leq k-l \\ 3 \mid k-l}} \left(2 \binom{2k-1}{l} - \binom{2k}{l} \right) \equiv \binom{2k/3^{\nu_3(k)} - 1}{k/3^{\nu_3(k)} - 1} \pmod{3}.$$

Apparently for $0 \leq k \leq 3^a - 1$,

$$\binom{3^a-1}{k} = \prod_{j=1}^k \left(\frac{3^a}{j} - 1 \right) \equiv (-1)^k \pmod{3}.$$

Hence noting that $a \geq 2$ and applying (1.1), we can get

$$\begin{aligned}
f(a) &\equiv \sum_{k=1}^{3^a} (-1)^{k-1} \binom{3^a-1}{k-1} \binom{2k/3^{\nu_3(k)}-1}{k/3^{\nu_3(k)}-1} \equiv \sum_{j=1}^a \sum_{\substack{1 \leq i < 3^{a-j} \\ 3 \nmid i}} \binom{2i-1}{i-1} \\
&= 1 + \frac{1}{2} \sum_{j=1}^{a-1} \left(\sum_{i=0}^{3^{a-j}-1} \binom{2i}{i} - \sum_{i=0}^{3^{a-j-1}-1} \binom{6i}{3i} \right) \\
&\equiv 1 + \frac{1}{2} \sum_{j=1}^{a-1} \left(\sum_{i=0}^{3^{a-j}-1} \binom{2i}{i} - \sum_{i=0}^{3^{a-j-1}-1} \binom{2i}{i} \right) \equiv 1 - \frac{1}{2} \equiv -1 \pmod{3}.
\end{aligned}$$

Finally, if $m = 4$, then we have

$$\begin{aligned}
\frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{(-1)^k}{4^k} \binom{3^a-1}{k} \binom{2k}{k} &= \sum_{k=1}^{3^a} \binom{3^a-1}{k-1} \frac{(-1)^{k-1}}{k} \sum_{l=0}^{k-1} \frac{1}{4^l} \binom{2l}{l} \\
&\equiv \sum_{k=1}^{3^a} \frac{1}{k} \sum_{l=0}^{k-1} \binom{2l}{l} \pmod{3}.
\end{aligned}$$

Clearly,

$$\sum_{l=0}^{k-1} \frac{1}{4^l} \binom{2l}{l} = \sum_{l=0}^{k-1} (-1)^l \binom{-1/2}{l} = (-1)^{k-1} \binom{-1/2-1}{k-1} = \frac{k}{2^{2k-1}} \binom{2k}{k}.$$

So using (1.1) again, we obtain that

$$\frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{(-1)^k}{4^k} \binom{3^a-1}{k} \binom{2k}{k} \equiv 2 \sum_{k=1}^{3^a} \binom{2k}{k} \equiv 2 \left(\binom{2 \cdot 3^a}{3^a} - 1 \right) \equiv -1 \pmod{3}.$$

4. PROOFS OF (1.7) AND (3.1)

The key of the proofs of (1.7) and (3.1) is the following identity.

Lemma 4.1.

$$\sum_{k=0}^n \binom{2k}{k} \binom{n}{k} (-x)^k = \frac{1}{4^n} \sum_{k=0}^n \binom{2j}{j} \binom{2(n-j)}{n-j} (1-4x)^k. \quad (4.1)$$

Proof.

$$\begin{aligned}
\sum_{k=0}^n \binom{2k}{k} \binom{n}{k} (-x)^k &= \sum_{k=0}^n \binom{-1/2}{k} \binom{n}{k} (4x - 1 + 1)^k \\
&= \sum_{k=0}^n \binom{-1/2}{k} \binom{n}{k} \sum_{j=0}^k \binom{k}{j} (4x - 1)^j \\
&= \sum_{j=0}^n \binom{-1/2}{j} (4x - 1)^j \sum_{k=j}^n \binom{-1/2 - j}{k - j} \binom{n}{n - k} \\
&= \sum_{j=0}^n \binom{-1/2}{j} \binom{n - 1/2 - j}{n - j} (4x - 1)^j \\
&= \frac{1}{4^n} \sum_{j=0}^n \binom{2j}{j} \binom{2(n - j)}{n - j} (1 - 4x)^j.
\end{aligned}$$

□

Substituting $x = 1$ in (4.1), we get

$$\sum_{k=0}^{n-1} (-1)^k \binom{2k}{k} \binom{n-1}{k} = \frac{1}{4^{n-1}} \sum_{k=0}^{n-1} (-3)^k \binom{2k}{k} \binom{2(n-1-k)}{n-1-k}. \quad (4.2)$$

Let $a = \nu_3(n)$. Obviously,

$$\sum_{k=0}^{n-1} (-3)^k \binom{2j}{j} \binom{2(n-1-k)}{n-1-k} \equiv \sum_{k=0}^{2a-2} (-3)^k \binom{2k}{k} \binom{2(n-1-k)}{n-1-k} \pmod{3^{2a-1}}.$$

When $a = 1$, writing $n = 3b$ with $3 \nmid b$, we have

$$\sum_{k=0}^0 (-3)^k \binom{2k}{k} \binom{2(n-1-k)}{n-1-k} = \binom{6b-2}{3b-1}$$

is divisible by 3. Below we only consider $a \geq 2$. Noting that

$$\binom{2(n-1)}{n-1} \frac{\binom{n-1}{k}^2}{\binom{2(n-1)}{2k}} = \binom{2k}{k} \binom{2(n-1-k)}{n-1-k},$$

we have

$$\sum_{k=0}^{n-1} (-1)^k \binom{2k}{k} \binom{n-1}{k} \equiv \frac{1}{4^{n-1}} \binom{2(n-1)}{n-1} \sum_{k=0}^{2a-2} (-3)^k \frac{\binom{n-1}{k}^2}{\binom{2(n-1)}{2k}} \pmod{3^{2a-1}}.$$

Since

$$\binom{2(n-1)}{n-1} = \frac{n}{2(2n-1)} \binom{2n}{n} \equiv 0 \pmod{3^a},$$

it suffices to prove that

$$\sum_{k=0}^{2a-2} (-3)^k \frac{\binom{n-1}{k}^2}{\binom{2(n-1)}{2k}} \equiv 0 \pmod{3^{a-1}}.$$

Clearly,

$$\frac{\binom{n-1}{k}^2}{\binom{2(n-1)}{2k}} = \frac{1}{2k+1} \cdot \frac{\prod_{j=1}^k (1 - n/j)^2}{\prod_{j=2}^{2k+1} (1 - 2n/j)}.$$

For $1 \leq k \leq 2a-2$ and $2 \leq j \leq 2k+1$, it is easy to check that

$$\nu_3(j) \leq a-1$$

and

$$\nu_3((2k+1)j) \leq k+1.$$

Hence for $2 \leq j \leq k$

$$\frac{3^k(1 - n/j)}{2k+1} = \frac{3^k}{2k+1} - \frac{3^k n}{(2k+1)j} \equiv \frac{3^k}{2k+1} \pmod{3^{a-1}}.$$

Similarly, for $1 \leq j \leq 2k+1$, we also have

$$\frac{3^k}{(2k+1)(1 - 2n/j)} \equiv \frac{3^k}{2k+1} \pmod{3^{a-1}},$$

since

$$\frac{1}{1 - 2n/j} = 1 + \frac{2n}{j} + \left(\frac{2n}{j}\right)^2 + \dots$$

over the rational 3-adic field \mathbb{Q}_3 . Thus we get

$$\sum_{k=0}^{2a-2} (-3)^k \frac{\binom{n-1}{k}^2}{\binom{2(n-1)}{2k}} \equiv \sum_{k=0}^{2a-2} \frac{(-3)^k}{2k+1} \pmod{3^{a-1}}.$$

Note that for $k \geq 2a-1$, we always have

$$\frac{3^k}{2k+1} \equiv 0 \pmod{3^{a-1}}.$$

Thus (3.1) immediately follows from the following lemma.

Lemma 4.2.

$$\sum_{k=0}^{\infty} \frac{(-3)^k}{2k+1}$$

vanishes over \mathbb{Q}_3 .

Proof. Let \mathbb{C}_3 denote the completion of the algebraic closure of \mathbb{Q}_3 . For any $x \in \mathbb{C}_3$ with the 3-adic norm $|x|_3 < 1$, define the 3-adic logarithm function

$$\log_3(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Clearly,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-3)^k}{2k+1} &= \frac{1}{\sqrt{-3}} \sum_{k=0}^{\infty} \frac{(\sqrt{-3})^{2k+1}}{2k+1} = \frac{1}{2\sqrt{-3}} \left(\sum_{k=1}^{\infty} \frac{(\sqrt{-3})^k}{k} - \sum_{k=1}^{\infty} \frac{(-\sqrt{-3})^k}{k} \right) \\ &= \frac{1}{2\sqrt{-3}} (\log_3(1 + \sqrt{-3}) - \log_3(1 - \sqrt{-3})) = \frac{1}{2\sqrt{-3}} \log_3 \left(\frac{\sqrt{-3} - 1}{2} \right). \end{aligned}$$

Since $(\sqrt{-3} - 1)/2$ is a third root of unity, the lemma is concluded. \square

The proof of (1.7) is very similar, only requiring a few additional discussions. Now we have

$$\sum_{k=0}^{3^a-1} (-1)^k \binom{2k}{k} \binom{3^a-1}{k} \equiv \frac{1}{4^{3^a-1}} \binom{2(3^a-1)}{3^a-1} \sum_{k=0}^{2a-1} (-3)^k \frac{\binom{3^a-1}{k}^2}{\binom{2(3^a-1)}{2k}} \pmod{3^{2a}}.$$

Since

$$\frac{1}{3^a} \binom{2(3^a-1)}{3^a-1} = \frac{1}{2(2 \cdot 3^a - 1)} \binom{2 \cdot 3^a}{3^a} \equiv -1 \pmod{3},$$

we only need to show that

$$\sum_{k=0}^{2a-1} (-3)^k \frac{\binom{3^a-1}{k}^2}{\binom{2(3^a-1)}{2k}} \equiv 3^{a-1} \pmod{3^a}.$$

Since $a \geq 2$, for $1 \leq k \leq 2a-1$ and $1 \leq j \leq 2k+1$, we always have $\nu_3(j) \leq a-1$, and we also have $\nu_3((2k+1)j) \leq k$ unless $k=1$ and $j=3$. Hence for $(k, j) \neq (1, 3)$,

$$\frac{3^k(1-n/j)}{2k+1} \equiv \frac{3^k}{2k+1} \pmod{3^a} \quad \text{and} \quad \frac{3^k}{(2k+1)(1-2n/j)} \equiv \frac{3^k}{2k+1} \pmod{3^a}.$$

That is, for $k \geq 2$,

$$\frac{\binom{3^a-1}{k}^2}{\binom{2(3^a-1)}{2k}} = \frac{1}{2k+1} \cdot \frac{\prod_{j=1}^k (1-n/j)^2}{\prod_{j=2}^{2k+1} (1-2n/j)} \equiv \frac{(-3)^k}{2k+1} \pmod{3^a}.$$

It follows that

$$\begin{aligned}
& \sum_{k=0}^{2a-1} (-3)^k \frac{\binom{3^a-1}{k}^2}{\binom{2(3^a-1)}{2k}} \equiv \sum_{k=0}^1 (-3)^k \frac{\binom{3^a-1}{k}^2}{\binom{2(3^a-1)}{2k}} + \sum_{k=2}^{2a-1} \frac{(-3)^k}{2k+1} \\
& \equiv 1 + (-3) \cdot \frac{\binom{3^a-1}{1}^2}{\binom{2(3^a-1)}{2}} = 1 - \frac{6(3^a-1)^2}{(2 \cdot 3^a - 2)(2 \cdot 3^a - 3)} \\
& = \frac{3^{a-1}}{1 - 2 \cdot 3^{a-1}} \equiv 3^{a-1} \pmod{3^a}.
\end{aligned}$$

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